

# Application of Unscented Kalman Filter with Non-symmetric Sigma Point Sampling on the Integrated Navigation System

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The integrations of a global positioning system (GPS) and an inertial navigation system (INS) usually use error state models with linear or non-linear Kalman filters. In high dynamic environments, these systems introduce errors in the system models due to linearization. To overcome this drawback, the unscented Kalman filter (UKF) is applied on the non-linear total state model of the GPS/INS integrated system because the UKF is a nonlinear estimator that is particularly well suited for complex nonlinear systems. The sigma points in UKF are usually sampled symmetrically around the mean value and the random variable to be transformed is assumed to be Gaussian. If the variables depart from Gaussian nature, the performance of the system is degraded. To enhance the UKF performance the non-symmetric sigma points sampling is addressed in this paper. The non-symmetric factors for sigma points are carefully chosen to avoid local and global sampling problems. The simulations are done using real INS and GPS data for symmetric UKF (SUKF) and non-symmetric UKF (NSUKF) algorithms. The navigation performance and robustness of the proposed algorithm are also compared with that of the SUKF. According to the simulation results from the application of NSUKF on a nonlinear total state model, the performance and robustness of the navigation system is significantly improved under the environment with a number of satellites less than four. Hence, NSUKF is better choice for low cost INS/GPS integrated navigation systems and a good alternative for SUKF.

## 1. Introduction

In a nonlinear GPS/INS navigation system, the system and the measurement models are simply linearized around the current state estimate to apply the Extended Kalman Filter (EKF). EKF techniques suffer from divergence during GPS outages when using low-cost IMUs due to approximations during linearization process and suboptimal modeling. The main reason is that the low-cost sensors have complex error characteristics which are stochastic in nature and difficult to model. To overcome these drawbacks and to enhance the performance and robustness of the tightly coupled integration, instead of an EKF, the UKF is applied on nonlinear model [1].

The UKF belongs to a bigger class of filters called Sigma-Point Kalman Filters or Linear Regression Kalman Filters, which are using the statistical linearization technique. This technique is used to linearize a nonlinear function of a random variable through a linear regression between  $N$  points drawn from the prior distribution of the random variables. Since we are considering the spread of the random variables the technique tends to be more accurate than Taylor series linearization. The spreading sigma points cover the mean and covariance of the variables [2].

The set of sigma points is carefully selected around the mean but the sampling problem arises out of the incorrect scaling of these points. To solve both global and local sampling problems the non-symmetric approach was proposed by K. Kim & C.G. Park [3]. The spreading of sigma points from one side of mean is fixed and that for other side is adjusted using the non-symmetric factor. Their algorithm is successfully applied on the in-flight alignment of an aerial vehicle with a large heading error.

Unscented filter also assumes conditional Gaussianity throughout the filter recursions and may lead to misleading results in case the density departs too far from the assumed Gaussian density [2][4]. In this

paper, we propose the UKF with two unequal non-symmetric distribution factors. Therefore, the set of sigma points can be scaled independently. The proposed algorithm is applied on the land integrated navigation system with GPS signal outages and the performance is evaluated in comparison with the symmetric UKF.

Section 2 introduces the principles of unscented transformation and section 3 describes incorporating concepts and method of proposed non-symmetric unscented transformation incorporation the state constraints. Models of tightly coupled GPS/INS navigation system and the computing steps of NSUKF are explained in section 4. Simulation of navigation algorithm with real navigation data and the comparison of performance between SUKF and NSUKF are described in section 5. Conclusions are reported in the last section.

## 2. Unscented Transformation

In unscented transformation, to calculate the mean and covariance of a random vector, a set of deterministically selected sigma-points which have the same mean and covariance as the original random vector is chosen. Each sigma-point is propagated through the nonlinear models, and the mean and the covariance of the transformed random vector is calculated from the propagated sigma-points. The weights  $\mathbf{W}^i$  associated with sigma points  $\mathbf{x}^i$  have to satisfy the following condition to provide an unbiased estimate. [3]

$$\sum_{i=0}^N \mathbf{W}^i = 1$$

The selected sigma points are propagated through the non-linear function  $f(\_)$ .

$$\mathbf{y}^i = f(\mathbf{x}^i)$$

Their mean and covariant are then calculated by

$$\bar{\mathbf{y}} = \sum_{i=0}^N \mathbf{W}^i \mathbf{y}^i, \quad \mathbf{P}_{yy} = \sum_{i=0}^N \mathbf{W}^i (\mathbf{y}^i - \bar{\mathbf{y}})^T$$

In symmetric sampling strategy,  $2n+1$  sigma points are symmetrically distributed about the mean  $\bar{\mathbf{x}}_k$  and defined as follows.

$$\mathbf{x}_k^0 = \bar{\mathbf{x}}_k, \quad \mathbf{x}_k^i = \bar{\mathbf{x}}_k + \gamma\sqrt{\mathbf{P}_k} \quad \text{for } i=1 \dots n, \quad \mathbf{x}_k^i = \bar{\mathbf{x}}_k - \gamma\sqrt{\mathbf{P}_k} \quad \text{for } i=n+1 \dots 2n$$

where  $\gamma = \sqrt{n + \lambda}$ , and  $\lambda = \alpha^2(n + \kappa) - n$  and it is a composite scaling parameter. The constant  $\alpha$  determines the spread of the sigma points around  $\bar{\mathbf{x}}_k$  and is usually set to a small positive value ( $0 \leq \alpha \leq 1$ ). The constant  $\kappa$  is a secondary scaling parameter which is usually set to 0 to  $3 - n$  and provides an extra degree of freedom for fine tuning of the higher order moments [2].

The weights associated with the sigma points are defined as

$$W_m^0 = \frac{\lambda}{n+\lambda}, \quad W_c^0 = \frac{\lambda}{n+\lambda} + (1 - \alpha^2 + \beta) \quad \text{and} \quad W_m^i = W_c^i = \frac{\lambda}{2(n+\lambda)} \quad i = 1 \dots 2n$$

## 3. Non-symmetric Sigma Point Sampling

The symmetric distribution of sigma points around the mean value is modified into non-symmetric form by introducing non-symmetric factors in sampling process. The  $2n+1$  sigma points are chosen as follows,

$$\left. \begin{aligned} \mathbf{x}_k^0 &= \bar{\mathbf{x}} \\ \mathbf{x}_k^i &= \bar{\mathbf{x}}_k + \mu_1 \gamma \sqrt{\mathbf{P}_k} \quad \text{for } i=1 \dots n \\ \mathbf{x}_k^i &= \bar{\mathbf{x}}_k - \mu_2 \gamma \sqrt{\mathbf{P}_k} \quad \text{for } i=n+1 \dots 2n \end{aligned} \right\} \quad (1)$$

where  $\mu_1$  and  $\mu_2$  are non-symmetric factors and  $\mu_1 \neq \mu_2$ .

The weights corresponding to the sigma points have to match exactly the expected mean and covariance. It must satisfy the following conditions

$$\sum_{i=0}^{2n} W^i = W_0 + n(W_1 + W_2) = 1,$$

$$\bar{x} = \sum_{i=0}^{2n} W^i \chi^i \text{ and}$$

$$P_{xx} = \sum_{i=0}^{2n} W^i (\chi^i - \bar{x})(\chi^i - \bar{x})^T$$

Substituting sigma points (given in Eqn. (1)) into above equations, we have

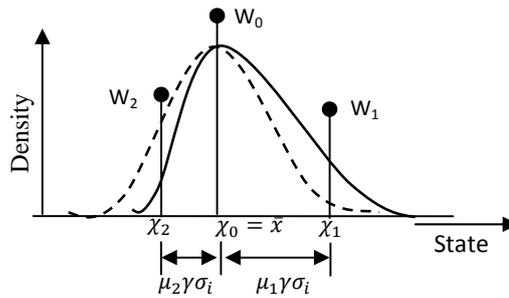
$$W_1 \mu_1 = W_2 \mu_2$$

$$(W_1 \mu_1^2 + W_2 \mu_2^2) \gamma^2 = 1.$$

The weights can be deduced as

$$W_1 = \frac{1}{(\mu_1^2 + \mu_1 \mu_2) \gamma^2}, W_2 = \frac{1}{(\mu_2^2 + \mu_1 \mu_2) \gamma^2}, W_0 = 1 - n(W_1 + W_2) \quad (2)$$

The proposed algorithm uses two degrees of freedom ( $\mu_1$  and  $\mu_2$ ) for scaling sigma points. These sigma points are propagated through the nonlinear function and mean and covariant are calculated using the scaling weights. Fig.1 shows the non-symmetric sigma points associated with the weights in 1 dimensional case. Dash curve is Gaussian probability density function (pdf) for symmetric unscented transformation and solid curve represents the non-Gaussian pdf for non-symmetric sampling. According to the nature of pdf, the distribution of sigma points is spread around the mean.  $\mu_1$  and  $\mu_2$  are needed to adjust to cover the stochastic nature.



**Fig.1** Non-symmetric sigma point sampling in 1 dimensional case  
( solid curve for a non-Gaussian pdf and dash curve for a Gaussian pdf)

#### 4. Tightly Coupled GPS/INS Navigation

The proposed NSUKF is applied on the tightly coupled integration scheme. INS data and GPS raw measurements (pseudorange and pseudorange rate) are processed in the data fusion algorithm, and the estimated errors are fed back to the INS to prevent the growth of navigation errors with time exhibited by an unaided INS[5]. The system dynamic models and the pseudorange and pseudorange rate measurement models are the key to the development of GPS/INS data fusion algorithms.

##### 4.1. System Models

The total state INS mechanization model is given by the following differential equations. In this work, navigation frame mechanization was chosen [5][6].

$$\left. \begin{aligned} \dot{L} &= \frac{v_N}{R_m+h}, \quad \dot{l} = \frac{v_E}{(R_l+h)\cos L}, \quad \dot{h} = -v_D \\ \dot{v}^n &= C_b^n f^b - (2\omega_{ie}^n + \omega_{en}^n) \times v^n + g^n \\ \text{The attitude quaternion is propagated by} \\ \dot{q} &= \frac{1}{2} \Omega(\omega_{nb}^b) q \end{aligned} \right\} \quad (3)$$

$$\text{where } \boldsymbol{\Omega}(\boldsymbol{\omega}_{nb}^b) = \begin{bmatrix} -[\boldsymbol{\omega}_{nb}^b \times] & \boldsymbol{\omega}_{nb}^b \\ -(\boldsymbol{\omega}_{nb}^b)^T & 0 \end{bmatrix},$$

$\boldsymbol{\omega}_{nb}^b$  is angular rate of a body frame relative to navigation frame and  $\mathbf{q} = [q_0 \ q_1 \ q_2 \ q_3]$  is the attitude quaternion vector.

The state vector is defined as

$$\mathbf{x} = [L \ l \ h : v_N \ v_E \ v_D : q_0 \ q_1 \ q_2 \ q_3 : \nabla_x \ \nabla_y \ \nabla_z : \varepsilon_x \ \varepsilon_y \ \varepsilon_z : cb \ d]^T$$

where  $\nabla = [\nabla_x \ \nabla_y \ \nabla_z]^T$  and  $\boldsymbol{\varepsilon} = [\varepsilon_x \ \varepsilon_y \ \varepsilon_z]^T$  are biases in accelerometer and gyro outputs, and  $\mathbf{v}_a$  and  $\mathbf{v}_g$  are white Gaussian noise corrupting the measurements. Latitude, longitude and height above the ellipsoid are denoted as  $L$ ,  $l$  and  $h$  and  $v_N$ ,  $v_E$  and  $v_D$  are velocities components in navigation frame respectively. The accelerometer bias and gyro bias are modeled as random constants and the clock bias  $cb$  in meter and clock drift  $d$  in meter per seconds of GPS receiver are calculated using random walk model [6].

#### 4.2. The Observation Models

The observables in our integration are pseudorange and pseudo-range rate of a GPS receiver. The pseudorange measurement  $\rho_i$  to the  $i^{\text{th}}$  satellite can be modeled as follows[6]:

$$\rho_i = \sqrt{(\mathbf{r}_i^n - \mathbf{C}_b^n \mathbf{l}_b)^T (\mathbf{r}_i^n - \mathbf{C}_b^n \mathbf{l}_b)} + cb + v_{\rho,i} \quad (4)$$

The satellite position in n-frame coordinates is denoted with  $\mathbf{r}_i^n$  while the position of the GPS antenna is  $\mathbf{C}_b^n \mathbf{l}_b$ .  $\mathbf{l}_b$  is the lever arm vector pointing from the origin of the body frame defined by the IMU to the GPS antenna. Additionally, the receiver clock bias  $cb$  and the measurement noise  $v_{\rho,i}$  are included. The pseudorange rate can be described by

$$\dot{\rho}_i = \mathbf{e}_i^n^T (\mathbf{v}_i^n - \mathbf{v}_{ins}^n - \mathbf{C}_b^n \boldsymbol{\omega}_{eb}^b \times \mathbf{l}_b) + d + v_{r,i} \quad (5)$$

$\mathbf{e}_i^n$  is the unit vector pointing from the GPS antenna to the  $i^{\text{th}}$  satellite and  $\boldsymbol{\omega}_{eb}^b$  the rate of body frame relative to the earth frame. Additionally, measurement noise  $v_{r,i}$  and the clock error drift  $d$  enter this observation model.

To apply the proposed NSUKF to the GPS/INS navigation system, the  $2n+1$  sigma points are generated according to the Eqn(1). They are propagated through the system models (Eqn(3)) and the mean and covariance are calculated using the weights shown in Eqn(2) as follows.

$$\begin{aligned} \bar{\mathbf{x}}_{k+1} &= \sum_{i=0}^{2n} \mathbf{W}_m^i \boldsymbol{\chi}_{k+1}^i \\ \mathbf{P}_{k+1}^{xx} &= \sum_{i=0}^{2n} \mathbf{W}^i (\boldsymbol{\chi}_{k+1}^i - \bar{\mathbf{x}}_{k+1}) (\boldsymbol{\chi}_{k+1}^i - \bar{\mathbf{x}}_{k+1})^T \end{aligned}$$

A set of predicted measurements is computed by propagating sigma points through the nonlinear measurement models given in Eqn(4) and Eqn (5).

$$\mathbf{y}_{k+1}^i = h(\boldsymbol{\chi}_{k+1}^i)$$

The predicted observation, innovation covariance and cross covariance are determined by

$$\begin{aligned} \bar{\mathbf{y}}_{k+1} &= \sum_{i=0}^{2n} \mathbf{W}_m^i \mathbf{y}_{k+1}^i \\ \mathbf{P}_{k+1}^{yy} &= \sum_{i=0}^{2n} \mathbf{W}_c^i (\mathbf{y}_{k+1}^i - \bar{\mathbf{y}}_{k+1}) (\mathbf{y}_{k+1}^i - \bar{\mathbf{y}}_{k+1})^T + \mathbf{R}_{k+1} \\ \mathbf{P}_{k+1}^{xy} &= \sum_{i=0}^{2n} \mathbf{W}_c^i (\boldsymbol{\chi}_{k+1}^i - \bar{\mathbf{x}}_{k+1}) (\mathbf{y}_{k+1}^i - \bar{\mathbf{y}}_{k+1})^T + \mathbf{R}_{k+1} \end{aligned}$$

Finally, the update can be performed by

$$\begin{aligned} \hat{\mathbf{x}}_{k+1}^+ &= \bar{\mathbf{x}}_{k+1} + \mathbf{K}_{k+1} (\mathbf{y}_{k+1} - \bar{\mathbf{y}}_{k+1}) \\ \mathbf{P}_{k+1} &= \mathbf{P}_{k+1}^{xx} - \mathbf{K}_{k+1} \mathbf{P}_{k+1}^{xy} \mathbf{K}_{k+1}^T \end{aligned}$$

where  $\mathbf{K}_{k+1} = \mathbf{P}_{k+1}^{xy} (\mathbf{P}_{k+1}^{yy})^{-1}$

## 5. Simulation Results

The trajectory data is collected using a low grade IMU and a GPS. The sampling rates are 100 Hz for IMU and 1Hz for GPS respectively. The differential GPS data is used as a reference data in our calculation. The numerical simulations are done in MATLAB using these data. First both SUKF and NSUKF are applied on the navigation system with fully GPS supported situation. There is no significant difference between the results and the trajectories obtained are given in Fig. 2. This is due to the less non-linearity in land navigations and small variation in vehicle dynamics during trajectory data collection.

To test the ability of the NSUKF algorithm on the GPS outages the satellite's data are rejected in data processing and each outage lasts for 60 seconds. The results under complete GPS outages are firstly analyzed and then followed by the results under partial GPS outages. Fig. 3, Fig. 4, Fig. 5 and Fig. 6 illustrate the 3D position errors over the 5 simulated outages in each case (i.e. for number of satellite visible equals 0, 1, 2, and 3). The benefit of more satellite availability can also be seen from these results. The general trend is that having three satellites visible is better than two, which is better than one and zero case. In each case the maximum position errors of NSUKF is less than that of SUKF. SUKF provides a minimum position error for the case of 3 satellites visible and NSUKF cannot reduce the error anymore. The main factor for this is the nonlinear capabilities of both UKFs which enabled the use of nonlinear system model as well as the nonlinear measurement model of the raw measurements without any need for approximations during linearization. Fig.7 shows that the comparison of the average maximum position errors between SUKF and NSUKF. It is found that NSUKF has superior performance than SUKF.

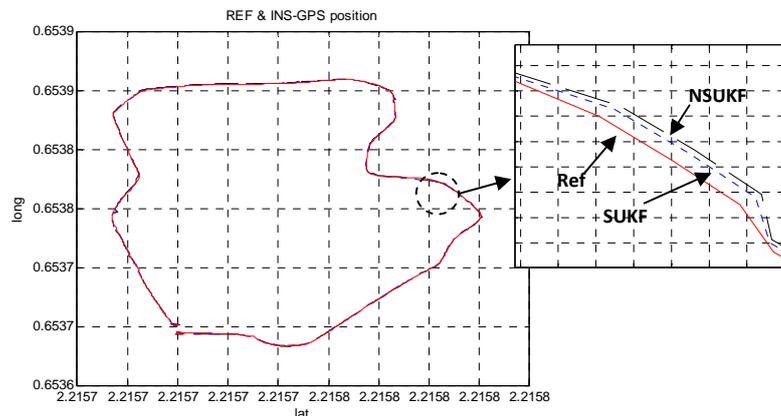


Fig. 2 Trajectories taken by SUKF and NSUKF solutions

## 6. Conclusion

In this paper, the non-symmetric unscented transformation with two degrees of freedom is proposed. The spreading of sigma points around the mean is adjusted carefully by two independent non-symmetric factors. The proposed algorithm is applied to the tightly coupled INS/GPS integration system and to avoid the linearization error, total state model of INS is used instead of error state model. The performance of the NSUKF is compared with the SUKF for the situations of GPS outages. It can be concluded that applying NSUKF on a nonlinear total state model does not degrade the performance of the navigation system significantly under the environment with less than four satellites. Non-symmetric sampling improves the navigation error and, the performance and robustness of the system is better than the symmetric case.

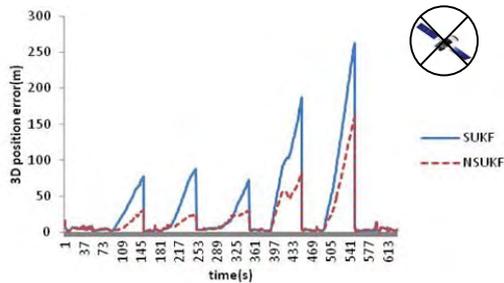


Fig. 3 Comparison of 3D position errors under complete GPS outages

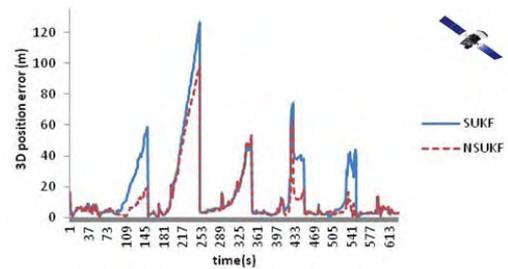


Fig. 4 Comparison of 3D position errors when 1 satellite is visible

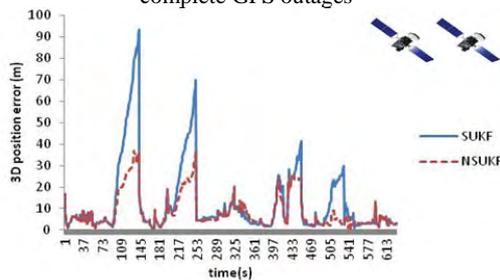


Fig. 5 Comparison of 3D position errors when 2 satellites are visible

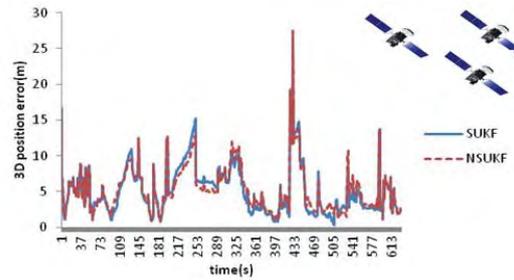


Fig. 6 Comparison of 3D position errors when 3 satellites are visible

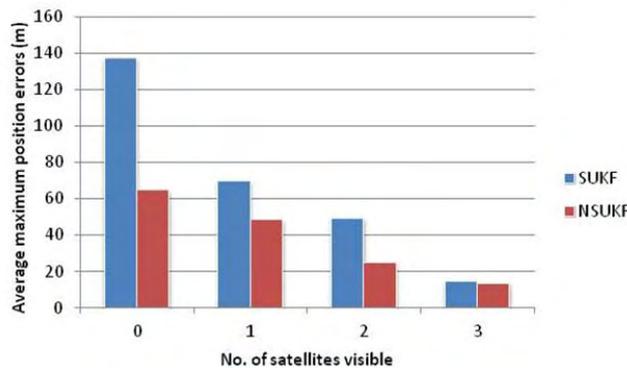


Fig.7 Average maximum position errors over the 5 outages with different numbers of satellites visible.

References

[1] X. Niu, S. Nasser, C. Goodalla1, and N. El-Sheimy, A Universal Approach for Processing any MEMS Inertial Sensor Configuration for Land-Vehicle Navigation. *Journal of Navigation*, 2007, vol. **60**, pp. 233-245.

[2] S. J. Julier, and J. K. Uhlmann, Unscented Filtering and Nonlinear Estimation, *Proceedings of the IEEE*, 2004, Vol . **92**, pp. 401-422.

[3] K.Kim and C.G. Park, Non-Symmetric Unscented Transformation with Application to In-Flight Aligment, *IJCAS*, vol. 8-4, pp. 776-781

[4] S. Sarkka, On Unscented Kalman Filtering for State Estimation of Continuous-Time Nonlinear Systems, *IEEE Tr. on Automatic Control*,2007, vol. **52**, pp. 1631-1641.

[5] D. H. Titterton, and J. L. Weston, *Strapdown Inertial Navigation Techonology*, 2<sup>nd</sup> Ed., IET and AIAA, 2005.

[6] Z. Syed, , P. Aggarwal, Y. Yang, and N. El-Sheimy, Improved Vehicle Navigation Using Aiding with Tightly Coupled Integration. *Proceedings of Vehicular Technology Conference*, Marina Bay, Singapore, 2008,pp. 3077-3081.

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